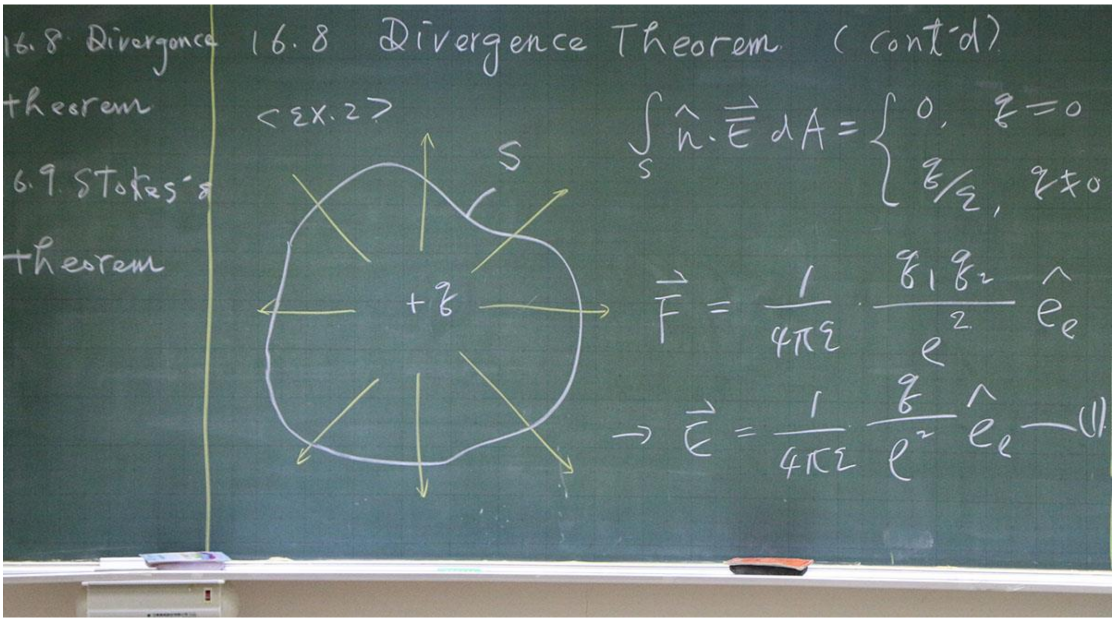

16.8 Divergence theorem
16.9 Stokes's theorem
16.8 Divergence Theorem (cont'd)
<EX.2>
S
+q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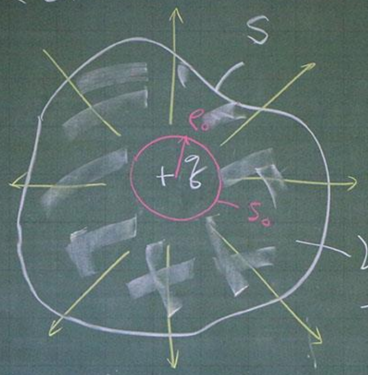
S
+q
$\int_S \hat{n}\cdot\vec{E}\,dA = \begin{cases} 0, & q=0 \\ q/\varepsilon, & q\neq 0 \end{cases}$
$\vec{F} = \frac{1}{4\pi\varepsilon}\cdot\frac{q_1 q_2}{e^2}\hat{e}_e$
$\rightarrow \vec{E} = \frac{1}{4\pi\varepsilon}\frac{q}{e^2}\hat{e}_e$ —(1)

16.8 Divergence theorem

16.9 Stokes's theorem

16.8 Divergence Theorem (cont'd)

<EX.2>



$$\int_S \hat{n}\cdot\vec{E}\,dA = \begin{cases} 0, & q=0 \\ q/\varepsilon, & q\neq 0 \end{cases}$$

$\mathcal{V}$, $\vec{F} = \dfrac{1}{4\pi\varepsilon}\cdot\dfrac{q_1 q_2}{e^2}\hat{e}_e$

$\rightarrow \vec{E} = \dfrac{1}{4\pi\varepsilon}\dfrac{q}{e^2}\hat{e}_e$ —— (1)

1. S does <u>NOT</u> enclose q (i.e., $q=0$)

$$\int_S \hat{n}\cdot\vec{E}\,dA = \int_{\mathcal{V}} \underline{\nabla}\cdot\vec{E}\,dV = 0 \text{ ——— (1)}$$

spherical coordinates.

2. S does enclose q!

$$\int_{\mathcal{V}'} \nabla\cdot\vec{E}\,dV = 0 = \boxed{\int_S \hat{n}\cdot\vec{E}\,dA} + \int_{S_0} \hat{n}\cdot\vec{E}\,dA$$

$$\int_S \hat{n}\cdot\vec{E}\,dA = -\int_{S_0} \hat{n}\cdot\vec{E}\,dA = -\int_{S_0} (-\hat{e}_\ell)$$

$$\cdot\left(\frac{1}{4\pi\varepsilon}\frac{q}{\ell_0^2}\right)\hat{e}_\ell\,dA = \int_{S_0} \frac{1}{4\pi\varepsilon}\frac{q}{\ell_0^2}\,dA$$

$$= \frac{q}{\cancel{4\pi}\varepsilon\cancel{\ell_0^2}} \times \cancel{4\pi}\cancel{\ell_0^2} = \frac{q}{\varepsilon} \quad \#$$

<u>PS</u>

$$\int_S \hat{n}\cdot\vec{E}\,dA = \int_V \nabla\cdot\vec{E}\,dV = \frac{q}{\varepsilon}$$

$$= \int_V \frac{q_V}{\varepsilon}\,dV$$ ∴ integrand should be the same $\rightarrow$ $\nabla\cdot\vec{E} = \dfrac{q_V}{\varepsilon}$ #

8. Divergence theorem

9. Stokes's theorem

<EX.3. Continuity eqn. of fluid mechanics>

$\vec{v}$

$$M = \sigma V$$

1. $$\frac{dM}{dt} = \frac{d}{dt}\int_V \sigma(x, y, z, t)\, dV$$

$$= \int_V \frac{\partial \sigma}{\partial t}\, dV \quad \text{——— (1)}$$

2. flow into V thru its surface.

$$-\int \underline{\sigma}\, \hat{n} \cdot \vec{v}\, dA \quad \text{——— (2)}$$

$$\frac{kg}{m^3} \cdot \frac{m}{sec} \cdot m^2 = \frac{kg}{sec}$$

3. mass generated by chemical reactions

$$\int_V f(x, y, z, t)\, dV \quad \text{——— (3)}$$

(1) = (2) + (3), we obtain,

$$\int_V \frac{\partial \sigma}{\partial t}\, dV = -\int_S \sigma\, \hat{n}\cdot\vec{v}\, dA + \int_V f\, dV \qquad \text{—— (4)}$$

$$+\int_S \sigma\, \hat{n}\cdot\vec{v}\, dA = \int_S \hat{n}\cdot(\sigma\vec{v})\, dA$$

$$= \int_V \nabla\cdot(\sigma\vec{v})\, dV \qquad \text{—— (5)}$$

$f\, dV$ —— (4)

—— (5)

Therefore, (4) becomes,

$$\int_V \left[\frac{\partial \sigma}{\partial t} + \nabla\cdot(\sigma\vec{v}) - f \right] dV = 0$$

$\because$ V is arbitrary $\therefore$ $[\ \ ] = 0$

$$\Rightarrow \frac{\partial \sigma}{\partial t} + \nabla\cdot(\sigma\vec{v}) - f = 0 \qquad \text{—— (6)}$$

8. Divergence theorem

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PS

1. There is no chemical reaction,

$$f=0 \rightarrow \frac{\partial \sigma}{\partial t} + \nabla \cdot (\sigma \vec{v}) = 0$$

2. $f=0$, incompressible field

$$\sigma = \text{const.} \quad \therefore \frac{\partial \sigma}{\partial t} = 0 , \quad \nabla \cdot (\sigma \vec{v})$$

$$= \cancel{\nabla \sigma}^{\,0} \cdot \vec{v} + \sigma \nabla \cdot \vec{v} \rightarrow \nabla \cdot \vec{v} = 0$$

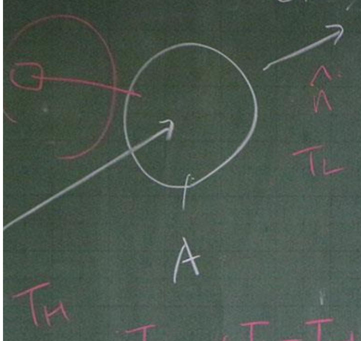
<Ex.4 Unsteady-state heat conduction>

1. $$H = mcT \rightarrow \frac{dH}{dt} = \frac{d}{dt} \int_V cT \sigma \, dV$$

$$= \int_V c \sigma \frac{\partial T}{\partial t} \, dV \quad \text{———— (1)}$$

2

$$Q \quad \frac{\text{Calories}}{\text{sec}} \rightarrow -K\,A\,\frac{\partial T}{\partial n} \qquad \text{Fourier conduction} \quad (2)$$



$$\rightarrow + \int_S \cdot \left(K \frac{\partial T}{\partial n}\right) dA$$

$$\frac{\partial T}{\partial n} = \frac{(T_L - T_H)}{\partial L}$$

3. chemical reaction, mechani-cal hysteresis, f

$$\int_V f(x, y, z, t)\, dV$$

(1) = (2) + (3) → conservation of Energy !!

$$\int_V c\sigma \frac{\partial T}{\partial t}\, dV = \int_S \hat{n} \cdot (K \nabla T)\, dA$$

$$K \frac{\partial T}{\partial n} = K \cdot (\nabla T \cdot \hat{n})$$

$$+ \int_V f\, dV \qquad (4)$$

where, $\int_S \hat{n} \cdot (K \nabla T)\, dA = \int_V \nabla \cdot (K \nabla T)\, dV$

$$\int_v \left[c\sigma \frac{\partial T}{\partial t} - \nabla\cdot(K\nabla T) - f \right] dV = 0 \qquad (5)$$

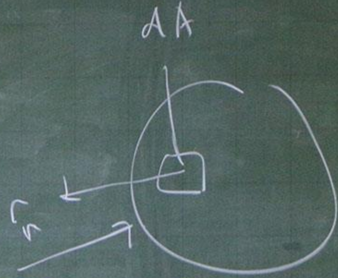
PS : $\dfrac{\partial T}{\partial n} = \nabla T \cdot \hat{n}$.

directional derivative

$$c\sigma \frac{\partial T}{\partial t} - K\nabla^2 T - f = 0$$

1. $f = 0$

$$c\sigma \frac{\partial T}{\partial t} - K\nabla^2 T = 0$$



$$\underbrace{\left(\frac{K}{c\sigma}\right)}_{\alpha^2}\nabla^2 T = 1\cdot\frac{\partial T}{\partial t} \Rightarrow \alpha^2\nabla^2 T = \frac{\partial T}{\partial t} \quad \#$$

<EX. 5>

$$J = -D\frac{\partial C}{\partial n} \Rightarrow \beta^2 \nabla^2 C = \frac{\partial C}{\partial t} \quad \#$$

Fick's 2nd law

Divergence
rem
Stokes's
rem

$$\alpha^2 \nabla^2 T = \frac{\partial T}{\partial t}$$

$\nearrow 0 \rightarrow$ steady state

$\searrow \neq 0 \rightarrow$ unsteady state

$\rightarrow \nabla^2 T = 0$ "Laplace Eqn."

ch. 18: diffusion eqn

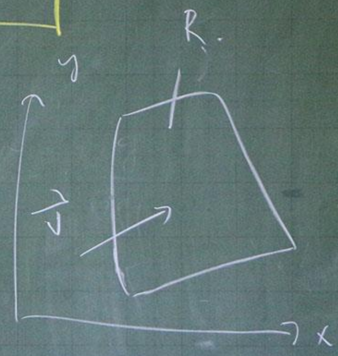
ch. 19: wave eqn, $c^2 \nabla^2 E = \frac{\partial^2 T}{\partial t^2}$

ch. 20: Laplace

<2D case of Divergence theorem>

$$\boxed{\int_V \nabla\cdot\vec{V}\, d\overline{V} = \int_S \hat{n}\cdot\vec{V}\, dA}$$

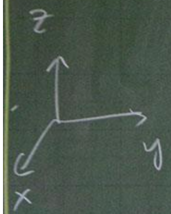
$$\vec{V} = V_x(x,y)\,\hat{i} + V_y(x,y)\,\hat{j}$$



<u>sol.</u>

$$dV = dA\,(1) = dA$$

$$dA = ds\,(1) = dS$$



<u>LHS of (*)</u>

$$\int_V \nabla\cdot\vec{V}\, d\overline{V} = \int_V \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + 0\right) dA$$

$$= \int_S \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y}\right) dA = \int_S \nabla\cdot\vec{V}\, dA \quad \text{—— (1)}$$

RHS of (*)

1. Top / bottom faces : $\hat{n}\cdot\vec{V} = 0$ $\left(\because \hat{n} = \pm\hat{k}\right)$

$$\rightarrow \quad \hat{k}\cdot\left(V_x\hat{i} + V_y\hat{j}\right) = 0$$

2. for the other 4 faces,

$$\int_S \hat{n}\cdot\vec{v}\,dA = \oint_C \hat{n}\cdot\vec{v}\,dS$$

$$\int_S \nabla\cdot\vec{v}\,dA = \oint_C \hat{n}\cdot\vec{v}\,ds$$

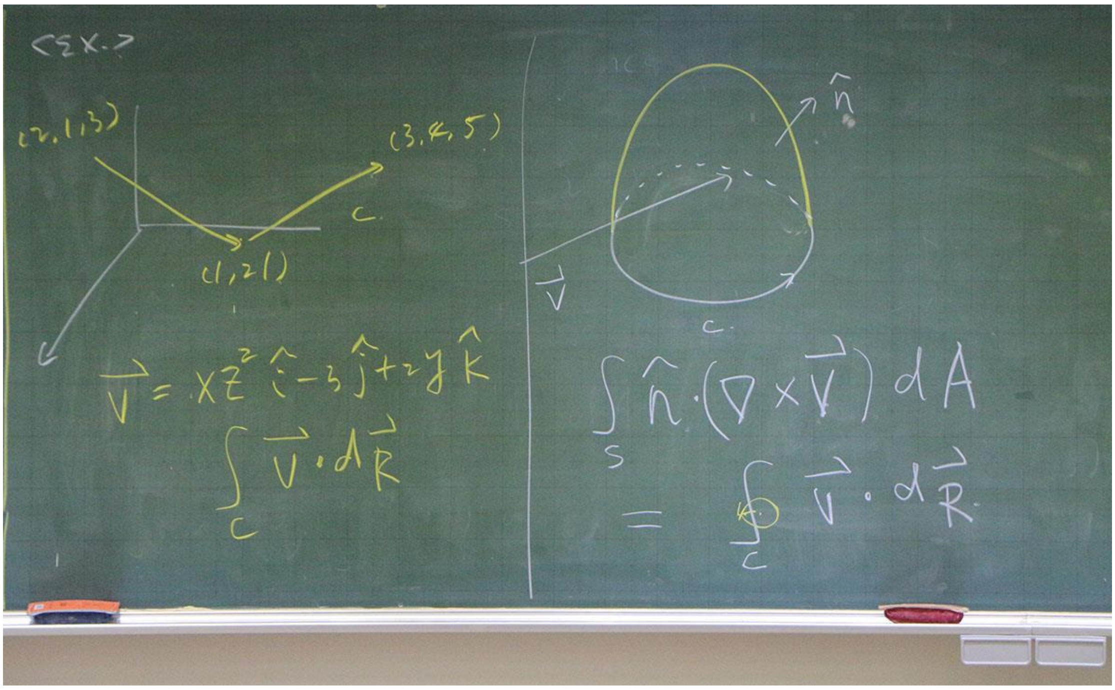
#

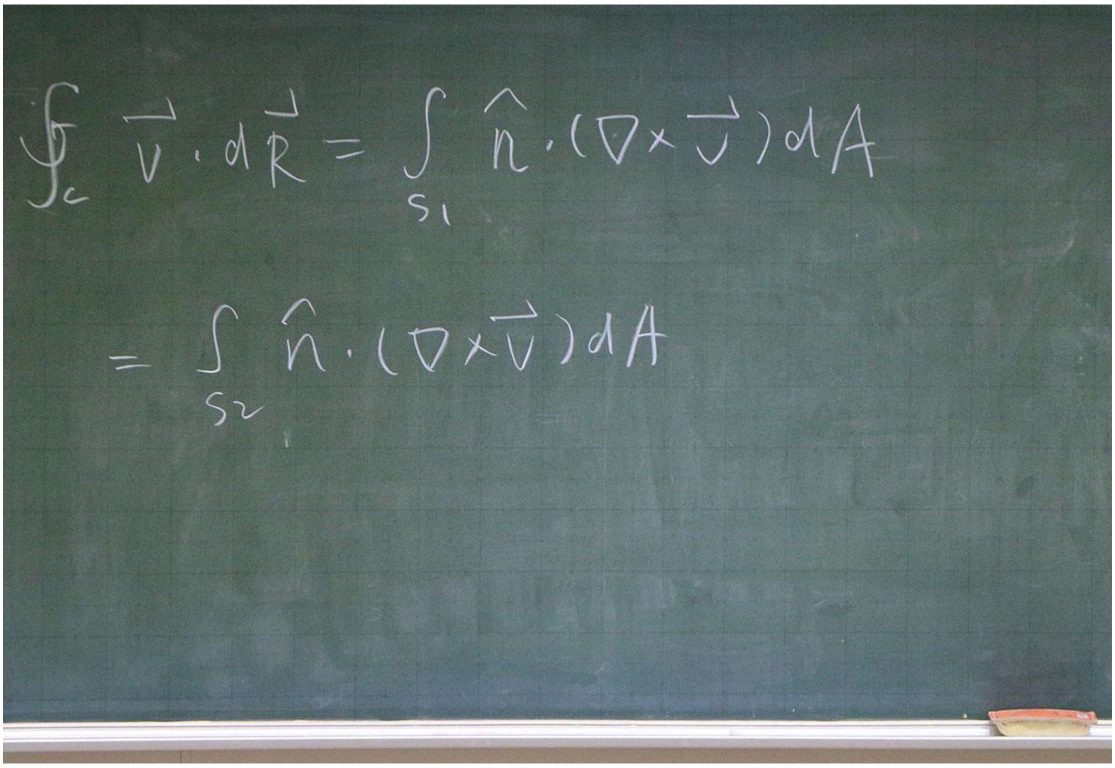
Divergence
eorem
Stokes's
eorem

16.9. Stokes's theorem

I. Background

$$\left\{\begin{array}{l} \int f\,dl = \int \vec{F}\cdot d\vec{R} \\ \int \vec{H}\cdot d\vec{R} = I \\ \int \vec{E}\cdot d\vec{R} \end{array}\right.$$


<EX.>
(2,1,3)
(3,4,5)
C
(1,2,1)
$\vec{V} = xz^2\hat{i} - 3\hat{j} + 2y\hat{k}$
$\int_C \vec{V}\cdot d\vec{R}$
$\hat{n}$
$\vec{V}$
C
$\int_S \hat{n}\cdot(\nabla\times\vec{V})\,dA$
$= \oint_C \vec{V}\cdot d\vec{R}$


$\oint_C \vec{V}\cdot d\vec{R} = \int_{S_1} \hat{n}\cdot(\nabla\times\vec{V})\,dA$
$= \int_{S_2} \hat{n}\cdot(\nabla\times\vec{V})\,dA$